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Transverse Vibration of a Beam Under Constant Stress

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Abstract

In this study, transverse vibrations of a beam made of two materials and with a variable cross section were investigated. Dimensionless natural frequency values of the system were found by the Rayleigh-Ritz approach. Moreover, the energy amounts of the system accumulated per unit mass were calculated. The results were given in tables for comparison.

Key Words: Transverse vibration, Dimensionless natural frequency, Isotropic material.

Sabit Gerilmeli Bir Kirişin Enine Titreşimlerinin İncelenmesi

Özet

Bu çalışmada değişken kesitli izotrop malzemeden yapılmış bir kirişin enine titreşimleri incelendi. Rayleigh-Ritz yaklaşımı kullanılarak, sistemin boyutsuz doğal frekans değerleri bulundu. Ayrıca sistemin birim kütle için biriktirdiği enerji miktarları hesaplandı. Değişik malzeme ve hız değerleri için simulasyonlar gerçekleştirildi. Karşılaştırma yapabilmek için sonuçlar tablolar halinde verildi.

Anahtar Sözcükler: Enine titreşimler, Boyutsuz doğal frekans, İzotrop malzeme

1. Introduction

Many investigations were carried out on vibrations of beams by Prescott (1961) and Meirovitch (1971). Thickness function for a constant stress beam made of isotropic material and subjected only to bending stress was obtained by Georgian (1989). A study for the optimum design of a flywheel was done by Georgian (1989). Natural frequencies about this flywheel were obtained for various modes by Güven and Çelik (1995), and Çelik (1988). The flywheel was considered to have a constant stress as the design criteria and the stress values about the flywheel were obtained. In this study, transverse vibrations of a rotating cantilever beam with a rectangular cross section under constant stress were investigated. The Rayleigh-Ritz method was used for analysis (Meirovitch, 1971). First of all, the thickness function and the stress equations satisfying the constant stress condition was obtained. To provide the boundary conditions, a mass was fixed to the beam. To the beam to accumulate maximum energy, the outer isotropic material should have high density and the inner material should have low density and high strength.

The aim of this study was to investigate the effect of designing the beam sections stressed uniformly under centrifugal force on lateral vibrations of beams.

2. Mathematical Model

2.1. Beam with constant stress

A variable crossbeam with constant stress consists of two parts, namely, the constant cross section part and the variable cross section part as can be seen in Figures 1 and 2.

The following equation can be obtained by force equilibrium

$$-\sigma_1 h(x) b_1 + P(x) + (\sigma_1 + d\sigma_1)(h + dh) b_1 = 0 (1)$$

Here, h, b_1, m, σ_1, and P are thickness fraction, constant cross section width, mass, radial stress and force respectively,

$$P(x) = m(x) x \Omega^2$$

$$m(x) = \rho_a b_1 h(x) dx$$

Second order terms i.e. in equation [2.1] are ignored and dividing by b_1 dx the following equation is obtained.

$$\frac{1}{h}\frac{dh}{dx} = -\frac{\rho_a \Omega^2}{\sigma_1} x \tag{2}$$

Thickness function can be derived by integration of equation [2.2] with boundary condition

$$x = a \text{ and } h = h_a$$

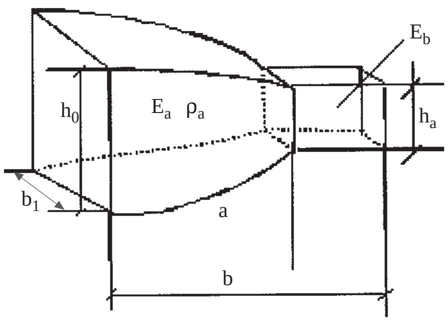

Figure 1. Beam with constant stress

$$h = h_a e^{\frac{\rho_a \Omega^2 a^2}{2\sigma_1}[1-(\frac{x}{a})^2]} \tag{3}$$

In equation [2.3], x=0 and h=h_0 are taken as boundary conditions, stress value can be found as follows.

$$\sigma_1 = \frac{\rho_a \Omega^2 a^2}{2ln\frac{h_0}{h_a}} \tag{4}$$

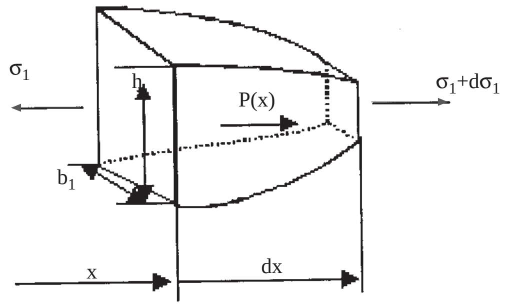

Figure 2. Variable cross section beam with constant stress

2.2. Beam with constant cross section

A constant cross section part is added to the beam in order to achieve the boundary condition in Figure 3.

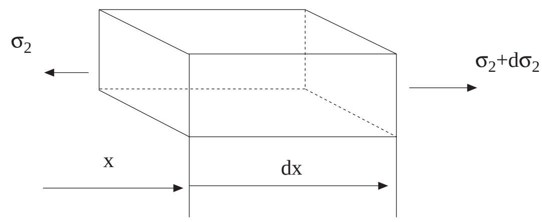

Figure 3. Beam with constant cross section

Using equilibrium at force effected on added part, the following equation is found:

$$-\sigma_2 A_b + \rho_b \Omega^2 A_b x dx + (\sigma_2 + d\sigma_2) A_b = 0 \tag{5}$$

Here, σ_2, A_b and h_a are radial stress, cross section area and thickness respectively
$A_b = b_1 h_a$

$$\sigma_2 = \frac{\rho_b \Omega^2 b^2}{2}\left[1 - (\frac{x}{b})^2\right] \tag{6}$$

Therefore, $\sigma_1 = \sigma_2$ at x=a

$$\frac{\rho_a \Omega^2 a^2}{2ln\frac{h_0}{h_a}} = \frac{\rho_b \Omega^2 b^2}{2}\left[1 - \left(\frac{a}{b}\right)^2\right] \tag{7}$$

A relation for a/b ratio can be found as follows for the optimum designed beam.

$$\frac{a}{b} = \left(\frac{1}{\frac{\rho_a}{\rho_b}\frac{1}{\frac{h_0}{h_a}} + 1}\right)^{\frac{1}{2}} \tag{8}$$

3. Energy Equations

Maximum potential and kinetic energy accumulated in the system are calculated. Transverse displacement can be calculated for harmonic vibration as follows:

$$w = W(x)\cos\omega_n t \tag{9}$$

W(x) is a function which provides geometric boundary conditions for transverse displacement. Maximum value of potential energy due to bending (V_{emax}) can be written as follows:

$$V_{emax} = \frac{E}{2}\int I(x)\left(\frac{\partial^2 W}{\partial x^2}\right)^2 dx \tag{10}$$

$$0 < x < a \rightarrow I(x) = \frac{b_1 h(x)^3}{12} = \frac{b_1 h_a^3}{12} e^{3ln\frac{h_0}{h_a}[1-(\frac{x}{a})^2]}$$

$$a < x < b \rightarrow I_b = \frac{b_1 h_a^3}{12}$$

Maximum potential energy is found for the whole beam due to bending as follows:

$$V_{emax} = \frac{E_a I_b}{2}\int_0^a \left(\frac{\partial^2 W}{\partial x^2}\right)^2 e^{3ln\frac{h_0}{h_a}[1-(\frac{x}{a})^2]}dx + \frac{E_b I_b}{2}\int_a^b \left(\frac{\partial^2 W}{\partial x^2}\right)^2 dx \tag{11}$$

Here,

E_a = Young elasticity module of first part

E_b = Young elasticity module of second part

Potential energy due to rotation (V_G) is written [2] as follows:

$$V_G = \frac{1}{2}\int\left(\frac{\partial w}{\partial x}\right)^2 \sigma A dx = \frac{1}{2}\int\left(\frac{\partial W}{\partial x}\right)^2 \sigma A\cos^2\omega_n t dx \tag{12}$$

Maximum potential energy is

$$V_{G_{max}} = \frac{1}{2}\int\left(\frac{\partial W}{\partial x}\right)^2 \sigma A dx \tag{13}$$

Total potential energy can be calculated by using the following equations:

$$V_{G_{max}} = \frac{b_1 h_a}{2}\int_0^a \left(\frac{\partial W}{\partial x}\right)^2 e^{ln\frac{h_0}{h_a}[1-(\frac{x}{a})^2]}\sigma_1 dx + \frac{b_1 h_a}{2}\int_a^b \left(\frac{\partial W}{\partial x}\right)^2 \sigma_2 dx$$

$$V_{G_{max}} = \frac{A_a}{2}\int_0^a \left(\frac{\partial W}{\partial x}\right)^2 e^{ln\frac{h_0}{h_a}[1-(\frac{x}{a})^2]}\sigma_1 dx + \frac{A_a}{2}\int_a^b \left(\frac{\partial W}{\partial x}\right)^2 \sigma_2 dx \tag{14}$$

Kinetic energy of the system can be found as follows:

$$dT = \frac{1}{2}\left(\frac{dw}{dt}\right)^2 dm = \frac{1}{2}(-\omega_n W\sin\omega_n t)^2 \rho A dx \tag{15}$$

$$T_{max} = \frac{1}{2}\rho\omega_n^2\int W^2 A dx$$

Maximum kinetic energy is

$$T_{max} = \frac{1}{2}\rho_a\omega_n^2 b_1 h_a\int_0^a W^2 e^{ln\frac{h_0}{h_a}[1-(\frac{x}{a})^2]}dx + \frac{1}{2}\rho_b\omega_n^2 b_1 h_a\int_a^b W^2 dx$$

$$T_{max} = \frac{A_a}{2}\rho_a\omega_n^2\int_0^a W^2 e^{ln\frac{h_0}{h_a}[1-(\frac{x}{a})^2]}dx + \frac{A_b}{2}\rho_b\omega_n^2\int_a^b W^2 dx \tag{16}$$

At the same time, kinetic energy can be rewritten as follows:

$$T = \frac{1}{2}J\Omega^2 \tag{17}$$

Mass inertia moment can be expressed as follows:

$$J = \int x^2 dm \qquad dm = \rho A dx$$

$$J = \rho_a b_1 h_a\int_0^a x^2 e^{ln\frac{h_0}{h_a}[1-(\frac{x}{a})^2]}dx + \rho_b b_1 h_a\int_a^b x^2 dx \tag{18}$$

Total mass m is,

$$m = \rho_a b_1 h_a\int_0^a e^{ln\frac{h_0}{h_a}[1-(\frac{x}{a})^2]}dx + \rho_b b_1 h_a\int_a^b dx \tag{19}$$

Inertia moment per unit mass is

$$\frac{T}{m} = \frac{\sigma_{1max}}{\rho_a}K \tag{20}$$

Here, K and are the shape function and maximum stress limit for constant stress area, respectively.

$$K = \frac{ln\frac{ho}{ha}}{(t)^2}\frac{\frac{\rho_a}{\rho_b}\int_0^2 \xi^2 e^{ln\frac{ho}{ha}[1-(\xi/t)^2]}.d\xi + \int_t^1 \xi^2 d\xi}{\frac{\rho_a}{\rho_b}\int_0^t e^{ln\frac{ho}{ha}[1-(\epsilon/t)^2]}.d\epsilon + \int_t^1 \epsilon d\epsilon} \qquad (21)$$

$$\xi = \frac{x}{b} \qquad t = \frac{a}{b},$$

Formulation of transverse displacement can be chosen as follows under geometric boundary conditions; the following matrix form can be obtained

$$W = a_0x^2 + a_1x^3 + a_2x^4 + \dots \qquad (22)$$

from solution of the energy equation by the Rayleigh-Ritz approach:

$$([A] - \omega_n^2[B])\begin{Bmatrix} a_0 \\ a_1 \\ . \\ . \\ a_n \end{Bmatrix} = 0 \qquad (23)$$

The non-dimensional natural frequencies can be found as follows:

$$\bar{\Omega} = \Omega\sqrt{\rho_b\frac{A_b b^4}{E_b I_b}} \qquad \bar{\omega}_n = \omega_n\sqrt{\rho_b\frac{A_b b^4}{E_b I_b}} \qquad (24)$$

4. Numerical Results

After the investigation of the vibrations about the beam with a variable cross section, dimensionless natural frequency values were obtained according to dimensionless rotating velocity, while dimensionless rotating velocity and $ln\frac{h_0}{h_a}$ value increase [Tables 1-3].

The outer part was made of high density and isotropical composite material [E_b=85 GPa] having high elasticity module. The frequency investigation was performed by changing the material [E_a=29-55-65GPa] of the inner part. While increasing the elasticity of the inner material, the frequencies increase. When comparing the frequencies of the constant cross section, the frequencies of the constant stress beam were observed to be higher [Tables 1,4]. When choosing a constant cross-sectioned outer part made of a material with high elasticity module and high density, and an inner part made of low density, the energy amounts of the constant stress beam accumulated per unit mass increases [Tables 5-7].

Non-dimensional natural frequency of beam with constant stress [Tables 1-3]

Table 1. E_a=29 GPa ρ_a=1550kg/m^3 E_b=85 GPa ρ_b=2000kg/m^3

$\bar{\Omega}$	$ln\frac{h_0}{h_a}$	$\bar{\omega}1$	$\bar{\omega}2$	$\bar{\omega}3$	$\bar{\omega}4$	$\bar{\omega}5$
0	0.5	3.92	21.9	63.77	135.9	240.7
	1	7.45	30.03	84.08	168.8	355.9
	1.5	13.5	42.76	111.1	234.4	549.2
	2	23.4	61.27	156.3	354.4	734.54
5	0.5	6.79	25.61	67.17	138.9	243.5
	1	9.45	32.46	86.22	170.7	357.4
	1.5	14.9	44.37	112.6	235.6	549.9
	2	24.4	62.36	157.2	355.1	735.0
10	0.5	11.5	34.38	80.51	147.4	252.0
	1	13.6	38.83	92.35	176.4	361.9
	1.5	18.2	48.87	116.9	239.1	558.2
	2	27.0	65.54	160.0	357.0	736.6

Table 2. E_a=55 GPa ρ_a=1550kg/m^3 E_b=85 GPa ρ_b=2000kg/m^3

$\bar{\Omega}$	$ln\frac{h_0}{h_a}$	$\bar{\omega}1$	$\bar{\omega}2$	$\bar{\omega}3$	$\bar{\omega}4$	$\bar{\omega}5$
0	0.5	5.23	26.5	73.8	151.5	285.2
	1	9.81	37.1	98.6	196.0	441.3
	1.5	17.6	52.6	133	286.9	649.7
	2	30.1	76.3	192	429.9	804.5
5	0.5	7.67	29.4	76.5	154.0	287.5
	1	11.4	39.0	100	197.6	442.5
	1.5	18.7	54.0	134.4	287.	650.3
	2	30.9	77.2	193.1	430.4	805.0
10	0.5	12.1	36.9	84.11	161.2	294.2
	1	15.1	44.3	105.4	202.3	445.8
	1.5	21.6	57.7	137.9	290.5	652.1
	2	33.2	79.7	195.3	432.0	806.4

Table 3. E_a=65 GPa ρ_a=1550kg/m^3 E_b=85 GPa ρ_b=2000kg/m^3

$\bar{\Omega}$	$ln\frac{h_0}{h_a}$	$\bar{\omega}1$	$\bar{\omega}2$	$\bar{\omega}3$	$\bar{\omega}4$	$\bar{\omega}5$
0	0.5	5.66	28.12	77.59	157.9	304.4
	1	10.6	39.81	104	208.2	474.2
	1.5	19.1	56.61	142.3	308.1	682.7
	2	32.7	82.41	207.0	457.9	836.5
5	0.5	7.99	30.93	80.13	160.3	306.5
	1	12.1	41.64	105.6	209.7	475.2
	1.5	20.1	57.85	143.4	309.0	683.2
	2	33.4	83.23	207.7	458.4	836.9
10	0.5	12.4	38.13	87.31	167.2	312.8
	1	15.7	46.69	110.4	214.2	478.3
	1.5	22.8	61.41	140.8	311.5	685.0
	2	35.5	85.64	209.7	460.0	838.3

While the $ln\frac{h_0}{h_a}$ ratio increases the energy amount per unit mass increases [Tables 5-11]. When a material with high strength and low density is used in the inner and outer part, the accumulated energy reaches the maximum value [Table 6].

Non-dimensional cantilever beam with constant stress

Table 4. E_a=E_b=29 GPa

Ω	$\bar{\omega}1$	$\bar{\omega}2$	$\bar{\omega}3$	$\bar{\omega}4$	$\bar{\omega}5$
0	3.51	21.99	61.59	127.92	222.59
5	6.44	25.40	65.11	131.2	225.67
10	11.19	33.58	74.58	140.54	234.75

Variation energy value per unit mass for beam with constant stress Tables 5-11

Table 5. E_a=29 GPa σ_{amax}=115 MPa E_b=85 GPa

$ln\frac{h_0}{h_a}$	K	T/M
0.5	0.38	11.08
1	0.44	12.75
1.5	0.47	13.95
2	0.48	13.95

Table 6. E_a=55 GPa σ_{amax}=480 MPa E_b=85 GPa

$ln\frac{h_0}{h_a}$	K	T/M
0.5	0.4	41.01
1	0.45	46.30
1.5	0.47	48.84
2	0.49	50.13

Table 7. E_a=65 GPa σ_{amax}=290 MPa E_b=85 GPa

$ln\frac{h_0}{h_a}$	K	T/M
0.5	0.4	24.78
1	0.45	27.97
1.5	0.47	29.51
2	0.49	30.29

Table 8. E_a=E_b=29 GPa σ_{amax}=115 MPa ρ_b=1550 kg/m^3

$ln\frac{h_0}{h_a}$	K	T/M
0.5	0.42	12.17
1	0.45	13.27
1.5	0.48	13.81
2	0.49	14.09

Table 9. E_a=E_b=55 GPa σ_{amax}=480 MPa ρ_b=1550 kg/m^3

$ln\frac{h_0}{h_a}$	K	T/M
0.5	0.42	43.6
1	0.46	47.53
1.5	0.48	49.46
2	0.49	50.46

Table 10. E_a=E_b=65 GPa σ_{amax}=290 MPa ρ_b=1550 kg/m^3

$ln\frac{h_0}{h_a}$	K	T/M
0.5	0.42	26.34
1	0.46	28.71
1.5	0.48	29.88
2	0.49	30.49

Table 11. E_a=E_b=85 GPa σ_{amax}=480 MPa ρ_b=2000 kg/m^3

$ln\frac{h_0}{h_a}$	K	T/M
0.5	0.42	33.79
1	0.46	36.83
1.5	0.48	38.33
2	0.49	39.11

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